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| Chapter 2 Calculus of exponential functions |

2.1 Overview

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Rates of change and the concept of a limit are fundamental ideas in the study of calculus. Many processes in nature can be modelled by exponential functions. These include the growth in a population of bacteria and the decay of radioactive material. Other areas modelled by exponential functions include the value of an investment after a period of time and the temperature of a liquid after it has been cooling.

In 1683, Jacob Bernoulli, a famous Swiss mathematician, was studying a problem relating to compound interest. What were the effects on the investment if smaller and smaller compounding intervals were used? In fact, he was attempting to find the value of limn→∞1+1nn and discovered that the limit had to lie between 2 and 3. Other mathematicians in the 17th century were studying similar ideas and came close to finding the limit.

Leonhard Euler, another famous Swiss mathematician, had studied under Bernoulli. While he was endeavouring to solve the problem of limits and investments proposed by Bernoulli, Euler discovered the constant e. He subsequently found many uses of the constant e and, in the mid 18th century published a paper showing all of his findings, including e to 18 decimal places. An approximation for e is 2.718 281 828 459 . . . Like π, Euler’s number, e, is irrational.

Euler studied many areas of science, including mechanics, fluid dynamics, astronomy and physics. In mathematics, the influence of Euler is found in geometry, trigonometry, calculus and algebra. He also studied and wrote on the theory of music.

LEARNING SEQUENCE

Overview

Review of limits and differentiation

The exponential function

Differentiation of exponential functions

Applications of exponential functions

Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at <www.jacplus.com.au>

Overmatter

The exponential function today plays a vital part in many areas of science, engineering and commerce. People analyse data through the use of mathematical models in order to increase our understanding of natural phenomena and to draw inferences about future behaviour. For example, scientists perform radiocarbon dating, ecologists study ecosystems calculating population growth, and accountants calculate investment options on financial data.

2.2 Review of limits and differentiation

2.2.1 Limits

The limit of a function, y=f (x), is the value that the function approaches as x approaches a given value.

Consider the limit of f (x)=x+1 as x approaches 1, using a spreadsheet.

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| x | x+1 |
| 0.9 | 1.9 |
| 0.99 | 1.99 |
| 0.999 | 1.999 |
| 0.9999 | 1.9999 |
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| x | x+1 |
| 1.1 | 2.1 |
| 1.01 | 2.01 |
| 1.001 | 2.001 |
| 1.0001 | 2.0001 |
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As x approaches 1 from the left-hand side, or below, the function approaches 2.

As x approaches 1 from the right-hand side, or above, the function approaches 2.

As both are equal, the limit exists and is written as:

limx→1 (x+1)=2

Since the function f(x)=x+1 is continuous, the limit can be found by direct substitution.

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Consider a different function, g(x)=x2−1x−1. This function is undefined at x=1.

However, g(x) can be simplified to g(x)=(x−1) (x+1) (x−1) .

g(x)=(x+1), x≠1

As shown above, since limx→1 (x+1)=2, then limx→1 x2−1x−1=2 or limx→1g(x)=2.

The graph of y=g(x) is in fact a linear function with a point discontinuity at (1, 2), as shown at right.

WORKED EXAMPLE 1

Evaluate the following limits:

limh→4(3h−5)

limx→0x2+5x+6x+2

limx→−2x2+5x+6x+2

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| THINK | | | WRITE |
| a. | 1. | Substitute h=4 as the function is defined. | limh→4(3h−5)  =3×4−5=7 |
|  | 2. | Answer the question. | limh→4(3h−5)=7 |
| b. | 1. | Substitute x=0 as the function exists for this value. | limx→0x2+5x+6x+2  =62=3 |
|  | 2. | Answer the question. | limx→0x2+5x+6x+2=3 |
| c. | 1. | The function is undefined at x=−2. Factorise the numerator and simplify. | limx→−2x2+5x+6x+2  =limx→−2(x+2) (x+3) (x+2) =limx→−2(x+3) |
|  | 2. | Substitute x=−2. | =−2+3=1 |
|  | 3. | Answer the question. | limx→−2x2+5x+6x+2=1 |
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2.2.2 The derivative as a limit

In Year 11, you were introduced to differentiation, the process of finding the rate of change of a function at any point.

Differentiation from first principles involves finding a limit as h approaches 0.

For the function y=f (x):

dydx=limh→0f (x+h)−f (x)h

WORKED EXAMPLE 2

Calculate the derivative of f(x)=3x2−4x+7 from first principles.

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| THINK | | WRITE |
| 1. | State the function | f (x)=3x2−4x+7 |
| 2. | The derivative is equal tolimh→0f (x+h)−f (x) h. | f ′(x)=limh→0f(x+h)−f(x) h |
| 3. | Substitute for f (x). | f ′(x)=limh→0[3(x+h) 2−4(x+h)+7]−[3x2−4x+7]h |
| 4. | Expand and simplify. | =limh→03x2+6xh+3h2−4x−4h+7−3x2+4x−7h=limh→06xh+3h2−4hh |
| 5. | Factorise and simplify. | =limh→0h(6x+3h−4) h=limh→0(6x+3h−4) |
| 6. | Evaluate the limit as h approaches 0. | =6x+3×0−4=6x−4 |
| 7. | Answer the question. | f ′(x)=6x−4 |
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2.2.3 Estimating a limit

Technology, such as a spreadsheet, can be used to estimate the limit of a given expression.

Consider the limit of ah−1h as h→0 for various values of a>0.

For a=2:

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| h | (2h−1)h |
| 1 | 1 |
| 0.5 | 0.828 427 12 |
| 0.1 | 0.717 734 63 |
| 0.01 | 0.695 555 01 |
| 0.001 | 0.693 387 46 |
| 0.000 1 | 0.693 171 2 |
| 0.000 01 | 0.693 149 58 |
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The limit is approaching 0.6931.

For a=3:

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| h | (3h−1)h |
| 1 | 2 |
| 0.5 | 1.464 101 615 |
| 0.1 | 1.161 231 74 |
| 0.01 | 1.104 669 194 |
| 0.001 | 1.099 215 984 |
| 0.000 1 | 1.098 672 638 |
| 0.000 01 | 1.098 618 323 |
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The limit is approaching 1.0986.

Can we find a value of a where the fraction ah−1h has a limiting value of 1 as h→0?

The value of a would lie in the interval 2<a<3.

If ah−1h=1, h≠0, then ah−1=h.

ah=1+ha(1+h)1h

Consider the value of a as h→0.

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| h | (1+h)1h |
| 1 | 2 |
| 0.5 | 2.25 |
| 0.1 | 2.593 742 46 |
| 0.01 | 2.704 813 829 |
| 0.001 | 2.716 923 932 |
| 0.000 1 | 2.718 145 927 |
| 0.000 01 | 2.718 268 237 |
| 0.000 001 | 2.718 280 469 |
| 0.000 000 1 | 2.718 281 694 |
| 0.000 000 01 | 2.718 281 786 |
| 0.000 000 001 | 2.718 282 052 |
| 1E−10 | 2.718 282 053 |
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The fraction is approaching 2.718 282 05… as h→0.

Euler’s number, e≈2.718 28, is the value of a that gives the limit of the fraction to be 1.

Like π, e is an irrational number.

Scientific and graphics calculators have an ex function that is treated in the same way as any other function.

An answer given in terms of e is an exact answer.

StudyON

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Limits and differentiation Summary screen and practice questions

Exercise 2.2

Review of limits and differentiation

WE 1a Evaluate the following limits.

limh→3(5h+4)

limh→−2(4−6h)

limh→0(6h2−3h+2)

WE 1b Evaluate the following limits.

limx→02x2+7x+3x−1

limx→2x2+4xx+2

limx→−3x2+4xx+1

WE 1c Evaluate the following limits.

limh→−3h2−h−12h+3

limh→0h2+4hh

limx→3x2−x−63−x

Evaluate the following limits.

limh→0(4x2+5xh−h2)

limh→03x2h+4h2h

WE 2 For the function f (x)=x2−6x, calculate the derivative from first principles.

Use first principles to differentiate the function f (x)=5+3x−2x2.

Estimate to 5 decimal places, using technology, the limit of ah−1h as h→0, where:

a=2.5

a=2.6

a=2.7

a=2.8

a=2.9

a=2.718 28

Evaluate the following, giving your answers to 4 decimal places.

e2

e3

e12

Evaluate the following, giving your answers to 3 decimal places.

2e−1

e3

4+ee

Use first principles to differentiate the function y=8x−x2.

Calculate the gradient of the tangent to the curve y=8x−x2  at the point where x=2.

Hence, determine the equation of the tangent to the curve at x=2.

Use differentiation by first principles to determine the gradient of the curve y=x3−3x2 at any point and hence the equation of the tangent at the point where the curve crosses the positive x-axis.

Consider the function f (x)=x3−4x.

Determine the x-intercepts of this function.

Use first principles to differentiate the function.

Calculate the gradient of the tangents at the points where the function crosses the x-axis.

Deduce that two of the tangents are parallel.

Simplify the fraction 1x+h−1x.

Hence, use first principles to determine the derivative of the function f (x)=1x, x≠0.

Show that 1(x+h−2)− 1(x−2)=−h(x−2)(x+h−2) .

Evaluate limh→0−1(x−2)(x+h−2) .

Hence, use first principles to determine the gradient function for y=1(x−2) , x≠2.

Determine the x-value(s) on the curve where the tangent is parallel to 9x+y−7=0.

2.3 The exponential function

2.3.1 Review of exponential functions, f(x)=ax where a∈R+ \ {1}

[MISSING IMAGE: c02uf003, c02uf003 ]

The graph of f (x)=ax, a >1 has the following features:

The y-intercept is (0, 1).

The key points are (1, a) and −1, 1a.

The maximal domain is x∈R.

The range is y∈ R+ or y>0.

As x→∞, f (x)→∞.

As x→−∞, f (x)→0.

The horizontal asymptote is y=0 (the x-axis).

It is a one-to-one function.

The graph of f (x)=ax for 0<a<1 can be written in the form f (x)=a−x, a>1. It is a reflection of f (x)=ax for a>1 over the y-axis.

[MISSING IMAGE: c02uf004, c02uf004 ]

The graph of f (x)=a−x, a>1 has the following features:

The y-intercept is (0, 1).

The key points is (−1, a) and 1, 1a.

The maximal domain is x∈R.

The range is y∈R+ or y>0.

As x→∞, f (x)→0.

As x→−∞, f (x)→∞.

The horizontal asymptote is y=0 (the x-axis).

It is a one-to-one function.

WORKED EXAMPLE 3

Sketch the following exponential functions, showing all important features.

 f(x)=2x

f(x)=2−x

f(x)=−2x

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| THINK | | | WRITE |
| a. | 1. | To sketch the graph of f (x)=2x, first determine the y-intercept, which occurs when x=0. | f (x)=2xf (0)=20f (0)=1  y-intercept (0, 1) |
|  | 2. | To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at x=±1. | f (1)=21f (1)=2 and  f (−1)=2−1f (−1)=12  (1, 2) and −1,12 |
|  | 3. | State the equation of the horizontal asymptote. | y=0 |
|  | 4. | Sketch the graph. | [MISSING IMAGE: , ] |
| b. | 1. | To sketch the graph of f (x)=2−x, first determine the y-intercept, which occurs when x=0. | f (x)=2−xf (0)=2−0f (0)=1  y-intercept (0,1) |
|  | 2. | To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at x=±1. | f (1)=2−1f (1)=12 and f (−1)=2−1f (−1)=2  1,12 and (−1,2) |
|  | 3. | State the equation of the horizontal asymptote. | y=0 |
|  | 4. | Sketch the graph. Note: This is a reflection of the curve in part a over the y-axis. | [MISSING IMAGE: , ] |
| c. | 1. | To sketch the graph of f (x)=−2x, first determine the y-intercept, which occurs when x=0. | f(x)=−2xf (0)=−20f (0)=−1  y-intercept (0,−1) |
|  | 2. | To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at x=±1. | f (1)=−21f (1)=−2 and  f (−1)=−2−1f (−1)=−12  (1,−2) and −1,−12 |
|  | 3. | State the equation of the horizontal asymptote. | y=0 |
|  | 4. | Sketch the graph. Note: This is a reflection of the curve in part a over the x-axis. | [MISSING IMAGE: , ] |
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2.3.2 The exponential function, f(x)=ex

It has been established that e≈2.718 28, giving 2<e<3.

Therefore, the graph of the exponential function, y=ex, lies between y=2x and y=3x.

[MISSING IMAGE: c02uf008, c02uf008 ]

The graph of f (x)=ex has the following features:

The y-intercept is (0, 1).

The key points are (1, e) and −1,1e.

The maximal domain is x∈R.

The range is y>0 or y∈R+.

As x→∞, f (x)→∞.

As x→−∞, f (x)→0.

The horizontal asymptote is y=0.

It is a one-to-one function.

The three exponential functions, y=2x, y=ex and y =3x are drawn at right.

Graphs of f (x)=aenx+b where a, n, b∈R can be sketched using your knowledge of transformations.

WORKED EXAMPLE 4

Sketch the function f (x)=2ex, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=2ex.

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| THINK | | | WRITE |
| a. | 1. | To sketch the graph of f (x) =2ex, first determine the y-intercept, which occurs when x=0. | f (x)=2exf (0)=2e0f (0)=2×1f (0)=2 y-intercept (0, 2) |
|  | 2. | To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at x=±1. | f (1)=2e1f (1)=2e and f (−1)=2e−1f (−1)=2e (1,2e)≈(1,5.44) −1,2e≈(−1,0.74) |
|  | 3. | State the equation of the horizontal asymptote. | y=0 |
|  | 4. | Sketch the function. | [MISSING IMAGE: c02uf009, c02uf009 ] |
| b. | State the transformation. | | The transformation required to map f (x)=ex onto f (x)=2ex is a dilation by a factor of 2 from the x-axis. |
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WORKED EXAMPLE 5

Sketch the function f(x)=2+e−x, showing all important features.

State the transformations required to map f(x)=ex onto f(x)=2+e−x.

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| THINK | | | WRITE |
| a. | 1. | To sketch the graph of f (x)=2+e−x, first determine the y-intercept, which occurs when x=0. | f (x)=2+e−xf (0)=2+e−0f (0)=2+1f (0)=3 y-intercept (0, 3) |
|  | 2. | To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at x=±1. | f (1)=2+e−1 and f (−1)=2+e−1f (−1)=2+e (1,2+e−1)≈(1,2.37) (−1,2+e)≈(−1,4.72) |
|  | 3. | State the equation of the horizontal asymptote. | y=2 |
|  | 4. | Sketch the function. | [MISSING IMAGE: c02uf010, c02uf010 ] |
| b. | State the transformations. | | The transformations required to map f (x)=ex onto f (x)=2+e−x are a reflection in the y-axis and a vertical translation upwards by 2 units. |
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2.3.3 Indicial equations with e.

Determining the x-intercepts of exponential functions may involve solving equations with e.

The laws of indices apply in the same way if e is the base.

Equations involving e are solved using the same methods as any equation involving indices.

Solving may require the use of logarithms with base e.

The laws of logarithms apply, using the notation loge⁡x or  ln x as found on your calculator.

Since ex>0, not all equations have real solutions. For example, ex=−1 has no real solution.

WORKED EXAMPLE 6

Consider the function f(x)=2−e−x.

Determine the coordinates of any axis intercepts.

Sketch the function f(x)=2−e−x, showing all important features.

State the transformations required to map f(x)=ex onto f(x)=2−e−x.

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| THINK | | | WRITE |
| a. | 1. | For the y-intercept, x=0.  Substitute x=0.  Evaluate. | f (x)=2−e−x  f (0)=2−e0  f (0)=2−1=1 |
|  | 2. | For the x-intercept, y=0.  Substitute y=0.  Rearrange the equation.  Take the log (base e) of both sides.  Use log laws to simplify. | 2−e−x=0  2=e−x  loge⁡2=loge⁡e−x  −xloge⁡e=loge⁡⁡2    x=−loge⁡2 |
|  | 3. | State the coordinates of the axis intercepts.  (Hint: An approximation may be useful.) | Axis intercepts: (−loge⁡⁡2,0) and (0,1) Approximately: (−0.693,0) and (0,1) |
| b. | 1. | To sketch the graph of f (x)=2−e−x, use the x- and y-intercepts found in part a. | Axis intercepts: (−loge⁡⁡2,0) and (0,1) Approximately: (−0.693,0) and (0,1) |
|  | 2. | To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at x=±1. | f (1)=2−e−1 and  f (−1)=2−e−1f (−1)=2−e (1,2−e−1)≈(1,1.632) (−1,2−e)≈(−1,−0.718) |
|  | 3. | State the equation of the horizontal asymptote. | y=2 |
|  | 4. | Sketch the function. | [MISSING IMAGE: , ] |
| c. | State the transformations. | | The transformations required to map f (x)=ex onto f (x)=2−e−x are reflections in both the x- and y-axes and a vertical translation upwards by 2 units. |
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WORKED EXAMPLE 7

Solve 6ex=15+ex for x, giving your answer:

as an exact number

correct to 3 decimal places.

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| THINK | | | WRITE |
| a. | 1. | Write the equation. | In6ex=15+ex |
|  | 2. | Collect like terms. | In5ex=15 |
|  | 3. | Find ex. | Inex=3 |
|  | 4. | Take the log of both sides. (Note: loge⁡x=lnx) | ln(ex)=ln3 |
|  | 5. | State the solution as an exact number. | In()x=ln3 |
| b. | 1. | Use your calculator to determine the approximation. | In()x=1.09861 |
|  | 2. | State the solution correct to 3 decimal places. | In()x=1.099 |
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WORKED EXAMPLE 8

Solve ex−3e−x=2 for x, giving your answer(s) correct to 2 decimal places.

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| THINK | | WRITE |
| 1. | Write the equation. | −003ex−3e−x=2 |
| 2. | Rewrite without negative indices. | −300ex−3ex=2 |
| 3. | Multiply each term by ex. | =000(ex)2−3=2ex |
| 4. | Recognise a quadratic equation in ex. | (ex)2−2ex−3=0 |
| 5. | Let a=ex. | 00a2−2a−3=0 |
| 6. | Factorise the quadratic expression. | (a−3)(a+1)=0 |
| 7. | Solve for a. | Inxa=3 or a=−1 |
| 8. | Substitute ex for a. | Inex=3 or 0ex=−1 |
| 9. | Solve for x. | lnex=ln3x=ln3, 0ex≠−1 |
| 10. | State the solution correct to 2 decimal places. | Inxx=1.10 (to 2 decimal places) |
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| Units 3&4 | Area 1 | Sequence 2 | Concept 2 |
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The exponential function Summary screen and practice questions

Exercise 2.3

The exponential function

WE 3 Sketch the following exponential functions, showing all important features.

f (x)=4x

f (x)=4−x

f (x)=−4x

On the same set of axes, sketch the graphs of y=10x and y=10−x.

WE 4 a. Sketch the function f (x)=4ex, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=4ex.

Sketch the function f (x)=−5ex, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=−5ex.

WE 5 a. Sketch the function f (x)=e−x+3, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=e−x+3.

Sketch the function f (x)=e2x+3, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=e2x+3.

WE 6 Consider the function f (x)=e2x−3.

Determine the coordinates of any axis intercepts for this function.

Sketch the function f (x)=e2x−3, showing all important features.

State the transformation required to map f(x)=ex onto f (x)=e2x−3.

Consider the function f (x)=4−2e−x.

Determine the coordinates of any axis intercepts for this function.

Sketch the function f (x)=4−2e−x, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=4−2e−x.

Sketch the function f (x)=4ex2, showing all important features.

State the transformation required to map f (x)=ex onto f (x)=4ex2.

Determine the coordinates of any axis intercepts for the function y=3e−x2−6.

Sketch the function, showing all important features.

State the transformation required to map y=ex onto y=3e−x2−6.

WE 7 Solve 3ex+8=5ex for x, giving your answer:

as an exact number

correct to 3 decimal places.

Solve for x, giving your answer correct to 3 decimal places.

ex=5

ex=12

ex=2.6

e−x=6

3=2ex

3e−x−10=0

Solve for x in each of the following, giving your answer in exact form.

(ex−1)(ex−2)=0

(ex−1)(ex+3)=0

(e−x−1)(e2x−4)=0

(3e−x−2)(2ex−1)=0

(2ex+1)(ex−4)=0

(3ex−2)(ex+4)=0

WE 8 Solve ex−15e−x=2 for x, giving your answer(s) correct to 2 decimal places.

Solve for x in each of the following, giving your answers in exact form.

5ex−12e−x−11=0

3ex+6e−x=11

2ex=9+5e−x

ex=25e−x

Solve for x in each of the following, giving your answers in exact form.

ex>1

e−x<e

e2x ≥4

e1−x ≤ 6

Sketch the curve y=2 e−x+1.

For what values of x is y<3?

Discuss why 2 e−x+1<0 has no real solutions.

Sketch the curve f(x)=4−ex, stating all axis intercepts in exact form.

For what values of x is y>0?

Discuss the range of the function if the domain is x≥0.

2.4 Differentiation of exponential functions

The derivative of the exponential function can be found using first principles.

If f (x)=ex, then

f ′(x)=limh→0f(x+h)−f(x)h=limh→0ex+h−exh=limh→0exeh−exh=limh→0ex(eh−1)h=exlimh→0eh−1h

Earlier, in subtopic 3.2, it was shown that limh→0ah−1h=1 for a=e; that is, limh→0eh−1h=1.

Substituting into f ′(x)=exlimh→0eh−1h:

f ′(x)=ex

The derivative of y=ef (x) can be found using the chain rule, studied in Year 11.

If y=ef (x) , let u=f (x) .

Then y=eu.

dydu= eu and  dudx=f ′(x)

The chain rule states:

dydx=dydu×dudx

Substitute:

dydx=eu×f ′(x)

Replace u as f (x):

dydx=f ′(x)ef(x)

d(ex)dx=ex

And with the chain rule:

d(ef(x))dx=f′(x) ef(x)d(eu)dx=eu×dudx where u=f(x)

WORKED EXAMPLE 9

Using the chain rule, differentiate y=e−5x with respect to x.

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| THINK | | WRITE |
| 1. | Write the equation. | dy=e−5x |
| 2. | Substitute u=−5x. | dy=eu and u=−5x |
| 3. | Determine dydu  and  dudx. | dydu=eu and dudx=−5 |
| 4. | Use the chain rule to find dydx. | dydx=dydu×dudx  dydx=eu×(−5) |
| 5. | State the derivative in terms of x. | dydx=−5ex |
|  | Alternatively, recognise and apply the formula d(ef (x) ) dx=f ′(x)ef(x) where f (x)=−5x. | y=e−5xdydx=−5ex |
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WORKED EXAMPLE 10

Determine the derivative of y=e2x+1.

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| THINK | | WRITE |
| 1. | Write the equation. | dy=e2x+1 |
| 2. | Recognise and apply the formula d(ef (x)) dx=f ′(x)ef (x)  wheref (x)=2x+1 | dydx=2e2x+1 |
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WORKED EXAMPLE 11

Differentiate the following.

f(x)=ex(ex−3)

f(x)=e3x−2e−xex

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| THINK | | | WRITE |
| a. | 1. | Write the equation. | ′f (x)=ex(ex−3) |
|  | 2. | Write the equation in expanded form. | ′f (x)=e2x−3ex |
|  | 3. | Differentiate each term. | f ′(x)=2e2x−3ex |
| b. | 1. | Write the equation. | ′f (x)=e3x−2e−xex |
|  | 2. | Separate the terms and simplify. | f (x)=e3xex−2e−xexf (x)=e2x−2e−2xf ′(x)=2e2x+4e−2x |
|  | 3. | Differentiate. | f ′(x)=2e2x+4e2x |
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WORKED EXAMPLE 12

Determine the derivative of the function y=ex3−x.

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| THINK | | WRITE |
| 1. | Write the equation. | =y=e(x3−x) |
| 2. | Substitute u=x3−x. | =y=eu and u=x3−x |
| 3. | Determine dydu  and  dudx. | dydu=eu and dudx=3x2−1 |
| 4. | Use the chain rule to find dydx. | dydx=dydu×dudx  dydx=eu×(3x2−1) |
| 5. | State the derivative in terms of x. | dydx=(3x2−1) e(x3−x) |
| Alternatively: | |  |
| 1. | Write the equation | =y=e(x3−x) |
| 2. | Recognise and apply the formulad(ef(x))dx=f ′(x)ef(x) where f(x)=x3−x | dydx=(3x2−1) e(x3−x) |
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Differentiation of exponential functions Summary screen and practice questions

Exercise 2.4

Differentiation of exponential functions

WE 9 Differentiate the following.

y=e10x

y=e13x

y=ex4

y=e−x

y=2e3x

y=4e−5x

WE 10 Differentiate the following.

y=e6x−2

y=e8−6x

y=2e5x+3

y=4e7−2x

y=−3e8x+1

y=−2e6−5x

y=10e6−9x

y=−5e3x+4

y=6e−7x

y=2ex2 + 1

y=3e2−x3

y=−4ex4+5

MC The derivative of y=e3x+2 is equal to:

3e3x+2

(3x+2)e3x+2

3e3x

3xe3x+2

WE 11 Differentiate the following

f (x)=2(ex+1)

f (x)=3e2x(ex+1)

f (x)=5(e−4x+2x)

f (x)=(ex+2)(e−x+3)

WE 11 Differentiate the following.

f (x)=3e3x+e−6xex

f (x)=4e7x−2e−xe−2x

WE 12 Determine the derivatives of the following.

y=ex2+3x

y=ex2−3x+1

y=ex2−2x

f(x)=e2−5x

Use the formula d(ef (x) )dx=f ′(x) ef (x) to differentiate the following functions.

f (x)=e6−3x+x2

g(x)=ex3+3x−2

h(x)=3e4x2−7x

y=−5e1−2x−3x2

MC The derivative of 6ex3−5x is equal to:

6(3x2−5)ex3−5x

(3x2−5)ex3−5x

6(x3−5x)ex3−5x

6(3x2−5)e3x2−5

If f (x)=5e9−4x, find the exact value of f ′(2).

If g(x)=2ex2−3x+2, find the exact value of g′(0).

Calculate the exact value of h′(−1) if h(x)=−5ex2+3x.

Find the equation of the tangent to the curve y=ex2+3x−4 at the point where x=1.

Determine the equations of the tangent and the line perpendicular to the curve y=e−3x−2 at the point where x=0.

Determine the derivative of the function f (x)=e−2x+3−4e and hence find:

f ′(−2) in exact form

{x:f ′(x)=−2}

Determine the derivative of the function f (x)=e3x+2ex and hence find:

f ′(1) in exact form

{x:f ′(x)=0}

2.5 Applications of exponential functions

Functions involving the exponential function y=ex can be used to model many real-life situations. These include:

population growth and decay, for example bacteria

radioactive decay

growth of investments

cooling of heated objects.

[MISSING IMAGE: c02uf012, c02uf012 ] [MISSING IMAGE: c02uf013, c02uf013 ]

In modelling that involves exponential growth and decay, the rate of change of the function often implies the change with respect to time, dydt.

It may be necessary to restrict the domain of the exponential function to suit the context of the problem.

WORKED EXAMPLE 13

The number of bacteria on a culture plate, N, can be defined by the rule

N(t)=2000e0.3t, t≥0

where t is the time, in seconds, the culture has been growing.

How many bacteria were initially present?

How many bacteria, to the nearest whole number, are present after 10 seconds?

At what rate is the bacteria population increasing after 10 seconds? Give your answer correct to the nearest whole number.

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| THINK | | | WRITE |
| a. | 1. | The initial time is when t=0. Substitute t=0 and evaluate. | N(t)=2000e0.3tN(0)=2000e0.3(0) =2000 |
|  | 2. | Write the answer. | Initially there were 2000 bacteria present. |
| b. | 1. | After 10 seconds, t=10. Substitute t=10. | N(t)=2000e0.3tN(10)=2000e0.3(10)=2000e3=40 171.074 |
|  | 2. | Write the answer. | After 10 seconds there were 40 171 bacteria present. |
| c. | 1. | Differentiate d(ef(x) )dx=f ′(x) ef(x) to find rate of change with respect to time. | N(t)=2000e0.3tdNdt=2000×0.3e0.3tdNdt=600e0.3t |
|  | 2. | After 10 seconds, t=10. Substitute t=10 and evaluate. | dNdt=600e0.3(10) =600e3=12 051.322 |
|  | 3. | Write the answer with correct units. | After 10 seconds the bacteria is increasing at a rate of 12 051 bacteria/second. |
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| TI | THINK | | WRITE | CASIO | THINK | | WRITE |
| a.1. | On a Calculator page, press MENU then select: 2: Add Graphs Complete the entry line in the f1(x)= tab as: 2000e0.3x | [MISSING IMAGE: c02uf014, c02uf014 ] | a.1. | On a Main Menu screen, select: Graph. Complete the entry line in the Y1 tab as: 2000e0.3x Note: The independent variable t has been replaced with x. | [MISSING IMAGE: c02uf024, c02uf024 ] |
| 2. | Sketch the graph by pressing the ENTER button. | [MISSING IMAGE: c02uf015, c02uf015 ] | 2. | Sketch the graph by pressing either the DRAW or EXE button. | [MISSING IMAGE: c02uf025, c02uf025 ] |
| 3. | To calculate the initial value, select: Menu 8: Geometry 1: Points & Lines 2: Point On. | [MISSING IMAGE: c02uf016, c02uf016 ] | 3. | To calculate the initial value, select: G-Solv (SHIFT F5) Y-CAL. Complete the entry line in X: as 0. | [MISSING IMAGE: c02uf026, c02uf026 ] |
| 4. | Move the cursor and select the curve representing f1x. Press the ESC (escape) button. This allows you to move the text box indicating the coordinate location of the point Px,y. Complete the entry line in the textbox as 0 for the x-value. | [MISSING IMAGE: c02uf017, c02uf017 ] | 4. | The answer appears on the screen. In this case, when x=0,  y=2000. | [MISSING IMAGE: c02uf027, c02uf027 ] |
| 5. | The answer appears on the screen. In this case, when x=0,  y=2000. | [MISSING IMAGE: c02uf018, c02uf018 ] | 5. | To calculate the number of bacteria after 10 seconds, select: G-Solv (SHIFT F5) Y-CAL. Complete the entry line in X: as 10. | [MISSING IMAGE: c02uf028, c02uf028 ] |
| b.1. | Complete the entry line in the textbox as 10 for the x-value. The answer appears on the screen. In this case, whenx=10, y=40171.074. Note: You can also calculate an x-value by entering a desired y-value. | [MISSING IMAGE: c02uf019, c02uf019 ] | b.1. | The answer appears on the screen. In this case, when x=10,  y=40171.074. | [MISSING IMAGE: c02uf029, c02uf029 ] |
| c.1. | On a Calculator page, press MENU then select: 4: Calculus 1: Numerical Derivative at a Point. | [MISSING IMAGE: c02uf020, c02uf020 ] | c.1. | On a Run Matrix screen, select: Math ddx. | [MISSING IMAGE: c02uf030, c02uf030 ] |
| 2. | Complete the value entry line as: 10 Press the OK button. | [MISSING IMAGE: c02uf021, c02uf021 ] | 2. | Complete the entry line as: ddx2000e0.3x Enter 10 in the x-value box. Press the EXE button to complete the calculation. | [MISSING IMAGE: c02uf031, c02uf031 ] |
| 3. | Complete the entry line as: ddx2000e0.3x Press the ENTER button to complete the calculation. | [MISSING IMAGE: c02uf022, c02uf022 ] | 3. | The answer appears on the screen. In this case, when x=10,dydx=12051.322. | [MISSING IMAGE: c02uf032, c02uf032 ] |
|  | The answer appears on the screen. In this case, when x=10,dydx=12051.322. | [MISSING IMAGE: c02uf023, c02uf023 ] |  |  |  |
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WORKED EXAMPLE 14

[MISSING IMAGE: c02uf033, c02uf033 ]

The mass, M grams, of a radioactive substance is initially 20 grams. After 30 years, the mass is 19.1 grams. The mass in any year is given by

M(t)=M0e−0.00152t

where M0 is a constant and t is the time in years.

Determine the value of M0.

Calculate the amount of the substance remaining after a further 30 years. Give your answer correct to 2 decimal places.

Determine the rate of decay at this time. Give your answer correct to 2 decimal places.

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| THINK | | | WRITE |
| a. | 1. | The mass, M grams, of a radioactive substance is initially 20 grams. Substitute t=0, M(0)=20. | 0M(t)=M0e−0.00152tM(0)=M0e−0.00152(0)=20 M0e0=20⇒M0=20 |
|  | 2. | Write the answer. | 0M0=20 |
| b. | 1. | Rewrite the equation with M0=20. | 0M(t)=20e−0.00152t |
|  | 2. | The mass, M grams, of a radioactive substance is initially 20 grams. After 30 years (t=30), the mass is 19.1 grams. A further 30 years is when t=60. Substitute t=60 and evaluate. | M(60)=20e−0.00152(60)=18.2567 |
|  | 3. | Write the answer. | After a further 30 years, the mass is 18.26 grams. |
| c. | 1. | Differentiate to find the rate of change with respect to time. | =M(t)=20e−0.00152tM′(t)=20×(−0.00152)e−0.00152tM′(t)=−0.0304e−0.00152t |
|  | 2. | Evaluate the rate of change at t=60 by substitution. | M′(60)=−0.0304e−0.00152(60) =−0.027 75 |
|  | 3. | Answer the question with correct units. | The rate of decay after 60 years is 0.03 grams/year. Note: The question asked for the rate of decay, so the negative sign is not included in the final answers. The negative indicates a rate that is decreasing. |
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WORKED EXAMPLE 15

[MISSING IMAGE: c02uf034, c02uf034 ]

The population of foxes on the outskirts of a city is starting to increase.

Data collected suggest that a model for the number of foxes is given by N(t)=480−320e−0.3t, t≥0, where N is the number of foxes t years after the observations began.

How many foxes were present initially at the start of the observations?

By how many had the population of foxes grown at the end of the first year of observations?

After how many months does the model predict the number of foxes would double its initial population?

Sketch the graph of N versus t.

Explain why this model does not predict that the population of foxes will grow to 600.

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| THINK | | | WRITE | |
| a. | Calculate the initial number. That is, when t=0. | | a. | 00N(t)=480−320e−0.3t When t=0, =N(0)=480−320e0=480−320=160 There were 160 foxes present initially. |
| b. | 1. | Calculate the number of foxes, N, after 1 year, t=1. | b. | When t=1, N(1)=480−320e−0.3≈242.94 After the first year 243 foxes were present. |
|  | 2. | Express the change over the first year in context. |  | Over the first year the population grew from 160 to 243, an increase of 83 foxes. |
| c. | 1. | Calculate the required value of t. Note: An algebraic method requiring logarithms has been used here. CAS technology could also be used to solve the equation. | c. | Let N=2×160=320.   320=480−320e−0.3t320e−0.3t=160 320e−0.3t=12−0.3t=loge12t=10.3loge12t≈2.31 |
|  | 2. | Answer the question. |  | 0.31×12≈4 The population doubles after 2 years and 4 months. |
| d. | Sketch the graph. | | d. | N(t)=480−320e−0.3t The horizontal asymptote is N=480. The y-intercept is (0,160).  [MISSING IMAGE: c02uf035, c02uf035 ] |
| e. | Give an explanation for the claim that this model will not predict the population of foxes to grow to 600. | | e. | The presence of an asymptote on the graph shows that as t→∞, N→480. Hence, N can never reach 600. The population will never exceed 480 according to this model. |
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Applications of exponential functions Summary screen and practice questions

Exercise 2.5

Applications of exponential functions

WE 13 The number of bacteria on a culture plate, N, can be defined by the rule

N(t)=500e0.46t, t≥0

where t is the time, in hours, that the culture has been growing.

How many bacteria were initially present?

How many bacteria, to the nearest whole number, are present after 5 hours?

At what rate is the bacteria population increasing after 5 hours? Give your answer correct to the nearest whole number.

The bilby is an endangered species that can be found in the Kimberley in Western Australia as well as some parts of South Australia, the Northern Territory and Queensland. The gestation time for a bilby is 2−3 weeks and when they are born, they are only about 11 mm in length. The growth of a typical bilby can be modelled by the rule

[MISSING IMAGE: c02uf036, c02uf036 ]

L=L0e0.599t

where L0 is its length in millimetres at birth and L is the length of the bilby in millimetres t months after its birth.

Determine the value of L0.

What is the rate of change of length of the bilby at time t months?

At what rate is the bilby growing when it is 3 months old? Give your answer correct to 3 decimal places.

WE 14 The mass, M grams, of a particular radioactive substance can be modelled by the exponential function

M(t)=M0e−0.005t

where M0 is a constant and t is the time in days, Initially the substance weighed 50 grams.

Determine the value of M0.

Calculate the amount of the substance remaining after 10 days. Give your answer correct to 2 decimal places.

Determine the rate of decay at this time. Give your answer correct to 2 decimal places.

Changing δ-gluconolactone into gluconic acid can be modelled by the equation y=y0e−0.6t where y is the number of grams of δ-gluconolactone present t hours after the process has begun. Suppose 200 grams of δ-gluconolactone is to be changed into gluconic acid.

Determine the value of y0.

How many grams of δ-gluconolactone will be present after 1 hour? Give your answer correct to the nearest gram.

How long will it take to reduce the amount of δ-gluconolactone to 50 grams? Give your answer correct to the nearest quarter of an hour.

Determine the rate of change in the δ-gluconolactone after 2 hours. Give your answer to 1 decimal place.

The decay of radon-222 gas is given by the equation y=y0e−0.18t, where y is the amount of radon remaining after t days. If initially there was 10 grams of radon-222 gas, determine:

the value of y0

the amount of gas, to the nearest integer, remaining after 2 days

how many days it will take for the mass to reach half its original mass

the rate of decay after 5 days, correct to 2 decimal places.

An amount of $1000 is invested in a building society where the 5% p.a. interest is compounded continuously. The amount in the account after t years can be modelled by the equation A= A0× ert, where A0 is the initial value of the investment and r is the continuous interest rate expressed as a decimal.

State the value of A0 and r.

Calculate the amount in the account, correct to the nearest cent, after:

1 year

5 years

10 years.

At what rate was the investment increasing, correct to the nearest cent per year, after:

1 year?

5 years?

10 years.

Estimate how long, to the nearest year, it would take for the investment to double in value.

A body that is at a higher temperature than its surroundings cools according to Newton’s Law of Cooling, which states that

T=T0e−zt

where T0 is the original excess of temperature, T is the excess of temperature in degrees Celsius after t minutes, and z is a constant.

The original temperature of the body was 95 °C and the temperature of the surroundings was 20 °C. Find the value of T0.

At what rate is the temperature decreasing after a quarter of an hour if it is known that z=0.034? Give your answer correct to 3 decimal places.

The number of people living in Boomerville at any time, t years, after the first settlers arrived can be modelled by the equation P= P0ekt. Initially, on 1 January 1850, Boomerville had a population of 500 people. By 1 January 1860, the population had grown to 675.

Determine the value of P0.

Calculate the value of k correct to 2 decimal places.

Using this value of k, determine the population on 1 January 1900. Give your answer to the nearest 10 people.

At what rate was the population increasing on 1 January 1900? Give your answer to the nearest whole number.

The mass, m kg, of a radioactive isotope remaining in a sample t hours after observations began is given by the rule m(t)=ae−kt. Initially there is 2 kg of the isotope. After 3 hours the mass of the isotope has decreased to 1.1 kg.

Determine the values of a and k. Give your answers correct to 1 decimal place where necessary.

Using these values, find the rate of change of the isotope as a function of t.

Calculate the rate of decay of the isotope after 6 hours. Give your answer to 2 decimal places.

Determine the half-life of this isotope, that is, the time it takes for the isotope to reduce to half its original mass. Give your answer correct to 1 decimal place.

An unstable gas decomposes in such a way that the amount present, A units, at time t minutes is given by the equation

A=A0e−kt

where k and A0 are constants. It was known that initially there were 120 units of unstable gas.

Find the value of A0.

After 2 minutes there were 90 units of the gas left. Find the value of k.

At what rate is the gas decomposing when t=5? Give your answer correct to 3 decimal places.

Will there ever be no gas left? Explain your answer.

The population of Australia since 1950 can be modelled by the rule

P=P0e0.016t

where P0 is the population in millions at the beginning of 1950 and P is the population in millions t years after 1950. It is known that there were 8.2 million people in Australia at the beginning of 1950.

Calculate the population in millions at the beginning of 2015, correct to 1 decimal place.

During which year and month does the population reach 20 million?

Determine the rate of change of population at the turn of the century, namely the year 2000, correct to 2 decimal places.

In which year is the rate of increase of the population predicted to exceed 400 000 people per year?

The pressure of the atmosphere, P cm of mercury, decreases with the height, h km above sea level, according to the law

P=P0e−kh

where P0 is the pressure of the atmosphere at sea level and k is a constant. At 500 m above sea level, the pressure is 66.7 cm of mercury, and at 1500 m above sea level, the pressure is 52.3 cm of mercury.

What are the values of P0 and k, correct to 2 decimal places?

Find the rate at which the pressure is falling when the height above sea level is 5 km. Give your answer correct to 2 decimal places.

The cane toad, originally from South America, is an invasive species in Australia. Cane toads were introduced to Australia from Hawaii in 1935 in an attempt to control cane beetles, though this proved to be ineffective.

[MISSING IMAGE: c02uf037, c02uf037 ]

In a controlled experiment at a particular waterhole, it was observed that at the beginning of the experiment there were an estimated 30 000 tadpoles (future cane toads) in the water. The number of tadpoles increased by about 60 000 a day over the period of a week. This growth pattern can be defined by the equation

T=T0ekt

where T0 is the initial number of cane toad tadpoles (in thousands) at the waterhole during the time of the experiment, T is the number of cane toad tadpoles (in thousands) at the waterhole t days into the experiment, and k is a constant.

Calculate the value of T0.

How many cane toad tadpoles are in the waterhole after a week if it is known that k=0.387? Give your answer to the nearest thousand.

Find the rate at which the cane toad tadpole numbers are increasing after 3 days.

WE 15 The population of possums in an inner city suburb is starting to increase. Observations of the numbers present suggest a model for the number of possums in the suburb given by P(t)=83−65e−0.2t, t≥0, where P is the number of possums observed and t is the time in months since observations began.

[MISSING IMAGE: c02uf038, c02uf038 ]

How many possums were present at the start of the observations?

By how many had the population of possums grown at the end of the first month of observations?

When does the model predict the number of possums would be twice the initial population?

Sketch the graph of P versus t.

Explain why this model does not predict the population of possums will grow to 100.

Manoj pours himself a mug of coffee but gets distracted by a phone call before he can drink the coffee. The temperature of the cooling mug of coffee is given by T=20+75e−0.062t, where T is the temperature of the coffee t minutes after it was initially poured into the mug.

What was the initial temperature of the coffee when it was first poured?

Sketch the graph of temperature, T °C against time, t minutes.

To what temperature will the coffee cool if left unattended?

How long does it take for the coffee to reach a temperature of 65°C? Give your answer correct to 2 decimal places.

Determine the rate of change in the temperature of the coffee after 10 minutes, correct to 1 decimal place. Explain why the rate of change is negative.

Newton’s Rule of Cooling states that the rate of change of the temperature of a particle is proportional to the difference between the temperature of the particle and the constant temperature of the surrounding medium. The temperature, T °C, of a particle when placed in a medium with a constant temperature of A °C can be modelled by the equation

T= T0e−kt+A

where t is time and T0 is a constant.

A metal ball has been heated to a temperature of 200 °C and is placed into a room that is maintained at a constant temperature of 30 °C. After 5 minutes, the temperature of the ball has dropped to 70 °C.

State the value of A and hence determine the value of T0.

Calculate the value of k correct to 4 decimal places.

Using the values found above, state the equation for this model and sketch its graph.

Determine the temperature of the rod, correct to 1 decimal place, after a further 10 minutes.

Calculate the rate of change in the temperature, correct to 1 decimal place, at this time.

Determine how long it will take for the temperature of the ball to reach 40 °C. Give your answer to 2 decimal places.

Verify, with reasons, that the metal ball would never reach 10 °C if left in the room.

Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

Evaluate the following limits.

limx→3(6x−1)

limx→32x2−6xx−3

limx→12x2+3x−5x2−1

limx→03x−52x−1

Using first principles, calculate dydx for the following functions.

y=4−x2

y=x2+4x

y=x(x+1)

Given f (x)=(x+5)2:

find f ′(x) using first principles

calculate f ′(−5) and explain its geometric meaning

calculate the gradient of the tangent to the curve y=f (x) at its y-intercept

calculate the instantaneous rate of change of the function y=f (x) at (−2, 9).

Solve the following for x, giving your answers to 3 decimal places.

ex+1=6

2e4−x−5=0

e−2x=8

4− ex−2=0

Solve the following for x, giving your answers in exact form.

e2x−2ex=0

(ex+1)(ex−3) =0

e2x+ 2ex=8

2e2x−9ex+4=0

Consider the function f(x)=−5x.

Evaluate f(2).

On the same set of axes sketch the graphs of y=5x, y=−5x and y=5−x.

Express y=5−x in an equivalent form.

Sketch the following graphs and state the domain and range of each graph.

y=2ex+1

y=3−3e−x2

y=−14ex+1

Sketch the following graphs and state the domain and range of each graph.

y=−2ex−3

y=4e−3x−4

y=5ex−2

Sketch the graph of y=2e1−3x−4, labelling any intercepts with the coordinate axes with their exact coordinates.

Sketch the graph of y=3×2x−24 and state its domain and range.

A possible equation for the graph shown is:

[MISSING IMAGE: c02uf039, c02uf039 ]

y=2−ex

y=2−e−x

y=2+e−x

y=e−x−2

Find the derivative of each of the following functions.

y=e−13x 

y=3x4−e−2x2

y=4ex−e−x+23e3x

y=(e3x−3)2

Consider the function defined by the rule f (x)=12e3x+e−x. Find the gradient of the curve when x=0.

Complex familiar

The graph shown is of the function f(x)=aex+b. Determine the values of a and b and write the function as a mapping.

[MISSING IMAGE: c02uf040, c02uf040 ]

The graph shown has an equation of the form y=Aenx+k. Determine its equation.

[MISSING IMAGE: c02uf041, c02uf041 ]

The graph of y=Ae−x2, where A is a constant, is shown.

[MISSING IMAGE: c02uf042, c02uf042 ]

Answer the following questions correct to 2 decimal places where appropriate.

If the gradient of the graph is 0 at the point (0, 5), determine the value of A.

Find dydx.

Determine the gradient of the tangent to the curve at the point where:

x=−0.5

x=1

Show that the equation of the tangent at the point where x=1 is given by 10x+ey−15=0.

The mass, m g, of a radioactive isotope remaining in a sample t hours after observations began is given by the rule m(t)=ae−kt. Initially there are 4 grams of the isotope. After 6 hours, the mass of the isotope has decreased to 2.8 g.

[MISSING IMAGE: c02uf043, c02uf043 ]

Evaluate the values of a and k. Give your answers correct to 3 decimal places.

Calculate the rate of decay of the isotope as a function of t.

Calculate the rate of decay after 6 hours. Give your answer correct to 2 decimal places.

The population of a certain town was 250 000 at the beginning of the year 2000 and 400 000 at the beginning of the year 2010. It was found that the relationship between the population, P thousands, and time, t years, could be modelled by the relationship P(t)=A ekt where A and k are constants.

[MISSING IMAGE: c02uf044, c02uf044 ]

Determine the values of A and k, giving your answers correct to 3 decimal places where necessary.

How many people lived in the coastal town at the beginning of the year 2015? Give your answer to the nearest thousand.

The local council has determined that the population should not exceed 750 000. In what year would this likely occur?

Complex unfamiliar

Microbiologists have been working with a certain type of bacteria that continues to thrive providing it has a favourable growth medium. For a particular experiment, they started with 500 bacteria and observed that the population doubles every 8 hours. The relationship between the number of bacteria, P, and the time, t hours since the bacteria started multiplying, is given by P(t)=P0 ekt, where P0 and k are constants.

[MISSING IMAGE: c02uf045, c02uf045 ]

State the value of P0 and show that k=18loge⁡2.

The growth phase lasts for 40 hours. How many bacteria are present in the colony at this time?

Show that the rate of increase in the colony size after 8 hours is 125loge⁡2 bacteria/hour.

Determine when the rate of increase in the colony size would be double the rate found after 8 hours.

The mass, m mg, of a radioactive substance remaining in a sample t hours after observations began is given by the rule m(t)=ae−kt. Initially, 30 mg of the substance is present. After 2 hours the scientist observes that 20% had disintegrated.

State the value of a.

Determine the mass of the substance after 2 hours.

Calculate the value of k, correct to 4 decimal places.

Determine the amount of the substance remaining after a further 3 hours. Give your answers correct to 2 decimal places.

Determine when the rate of decay of the substance is 1 mg/hour. Give your answer correct to 1 decimal place.

When money is invested in a bank at a constant rate of r% with continuously compounding interest, the accumulated amount $A at a time t years after the start of the investment is modelled by the equation

A=A0ert

where A0 is the amount to be invested.

[MISSING IMAGE: c02uf046, c02uf046 ]

Determine the amount to which $10 000 will grow in 6 years if invested at 4.5%.

If the rate did not change, estimate the time it would take for the investment to triple in value. Give your answer to the nearest month.

Newton’s Rule of Cooling states that the rate of change of the temperature of a particle is proportional to the difference between the temperature of the particle and the constant temperature of the surrounding medium. The temperature, T °C, of a particle when placed in a medium with a constant temperature of A °C can then be modelled by the equation

T=T0e−kt+A

where t is time and T0 is a constant.

A saucepan is filled with water and heated until the temperature of the water reaches 100 °C. The saucepan is then placed on a bench in a room that is kept at a constant temperature of 28 °C.

[MISSING IMAGE: c02uf047, c02uf047 ]

After 3 minutes, the temperature of the water in the saucepan is 76 °C.

State the value of A and determine the value of T0.

Show that k=13loge⁡32.

State the equation for this model and sketch a graph to represent it.

Determine the temperature of the water after a further 3 minutes.

Verify, with reasons, that the water would never reach 25 °C if left in the room.

StudyON

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Sit exam

Answers

2 Calculus of exponential functions

Exercise 2.2 Review of limits and differentiation

19

16

0

−3

3

32

−7

4

−2

4x2

3x2

f′(x)=2x−6

f′(x)=3−4x

0.916 29

0.955 51

0.993 25

1.029 62

1.064 71

1.000 00

7.3891

20.0855

1.6487

0.736

1.396

2.472

dydx=8−2x

4

y=4x+4

dydx=3x2−6x; y= 9x−27

(−2, 0), (0, 0), (2, 0)

dydx=3x2−4

At x=−2, gradient=8.

At x=0, gradient=−4.

At x=2, gradient=8.

The gradient of the tangent at x= −2 and x=2 is m=8. Therefore, since the gradients are equal, the tangents are parallel.

−h(x+h)x

dydx=−1x2

Sample responses can be found in the worked solutions in the online resources.

−1(x−2)2

dydx=−1(x−2)2

x=53, 73

Exercise 2.3 The exponential function

[MISSING IMAGE: an02uf001, an02uf001 ]

[MISSING IMAGE: an02uf002, an02uf002 ]

[MISSING IMAGE: an02uf003, an02uf003 ]

[MISSING IMAGE: an02uf004, an02uf004 ]

[MISSING IMAGE: an02uf005, an02uf005 ]

The function f(x)=ex has been dilated by a factor of 4 from the x-axis to give f(x)=4ex.

[MISSING IMAGE: an02uf006, an02uf006 ]

The function f (x)=ex has been reflected in the x-axis and dilated by a factor of 5 from the x-axis to give f (x)=−5ex.

[MISSING IMAGE: an02uf007, an02uf007 ]

The function f (x)=ex has been reflected in the y-axis and translated vertically up by 3 units to give f (x)=e−x+3.

[MISSING IMAGE: an02uf008, an02uf008 ]

The function f (x)=ex has been dilated by a factor of 12 from the y-axis and translated vertically up by 3 units to give f (x)=e−x+3.

Intercepts: 12loge⁡3, 0 and (0, −2)

[MISSING IMAGE: an02uf009, an02uf009 ]

The function f(x)=ex has been dilated by a factor of 12 from the y-axis and translated vertically down by 3 units to give f(x)=e−x−3.

Intercepts: (−loge⁡2,0) and (0, 2)

[MISSING IMAGE: an02uf010, an02uf010 ]

The function f (x)=ex has been dilated by a factor of 2 from the x-axis, reflected in the x-axis, reflected in the y-axis and translated vertically up by 4 units to give f (x)=4−2e−x.

[MISSING IMAGE: an02uf011, an02uf011 ]

The function f (x)=ex has been dilated by a factor of 2 from the y-axis and dilated by a factor of 4 from the x-axis to give f (x)=4ex2.

Intercepts: (−loge⁡2, 0) and (0,  2)

[MISSING IMAGE: an02uf012, an02uf012 ]

The function y= ex has been dilated by a factor of 2 from the y-axis and dilated by a factor of 3 from the x-axis, reflected in the y-axis and translated vertically down by 6 units to give y= 3e−x2−6.

x=ln 4

x=1.386

x≅1.609 

x≅−0.693

x≅0.956

x≅−1.792

x≅0.405

x≅−1.204

x=0, ln2

x=0

x=0, ln2

x=ln32, −ln2

x=ln4

x=(ln2−ln3)

x=1.61

x=ln 3

x=ln 3, ln23

x=ln 5

x=ln 5

x>0

x>−1

x≥ln2

x ≥1−ln6

[MISSING IMAGE: an02uf013, an02uf013 ]

x>0

For 2 e−x+1<0, the curve would be below the x-axis. But y>1 for all x values, so the curve is never below the x-axis. Hence, 2 e−x+1<0 has no real solutions.

[MISSING IMAGE: an02uf014, an02uf014 ]

x<ln4

For x=0, y=3.

Observing the graph:

For x≥0, y≤3.

If the domain is restricted to x≥0, the range is y ≤3 or y∈−∞, 3.

Exercise 2.4 Differentiation of exponential functions

10e10x

13ex3

14ex4

−e−x

6e3x

−20e−5x

6e6x−2

−6e8−6x

10e5x+3

−8e7−2x

−24e8x+1

10e6−5x

−90e6−9x

−15e3x+4

−42e−7x

ex2+1

−e2−x3

−ex4+5

A

2ex

3e2x(3ex+2)

−10(2e−4x−1)

3ex−2e−x

6e2x+7e−7x

36e9x−2ex

(2x+3)ex2+3x

(2x−3)ex2−3x+1

2(x−1)ex2−2x

−5e2−5x

(2x−3)e6−3x+x2

3(x2+1)ex3+3x−2

3(8x−7)e4x2−7x

10(1+3x)e1−2x−3x2

A

−20e

−6e2

−5e−2

y=5x−4

Tangent: y=−3x−1; perpendicular: y=13x−1

−2e7

32

2e2−2e

0

Exercise 2.5 Applications of exponential functions

500 bacteria

4987 bacteria

2294 bacteria/hour

11

dLdt=6.589e0.599t

39.742 mm/month

50

47.56 grams

0.24 grams/day

200

110 grams

214 hours

−11.8 grams/hour

10

7 grams

4 days

0.73 grams/day

A0=1000; r=0.05

$1051.27

$1284.03

$1648.72

$52.56/year

$64.20/year

$82.44/year

14 years

75

1.531 °C/min

500

0.03

2240

67 people/year

a=2; k= 0.2

dmdt=−0.4e−0.2t

0.12 kg/h

3.5 h

120

12loge⁡43

8.408 units/min

As t→∞, A→0. Technically the graph approaches the line A=0 (asymptotic behaviour). However, the value of A would be so small that in effect, after a long period of time, there is no gas left.

23.2 million

September 2005

0.29 million/year

2019

75.32 cm; k=0.24

5.45 cm of mercury/km

30

450 000 tadpoles

37 072.2 tadpoles/day

18 possums

12 possums

1.62 months

[MISSING IMAGE: an02uf015, an02uf015 ]

The presence of the asymptote at P=83 shows that as t→∞, P→83. The population can never exceed 83, so the population cannot grow to 100 possums.

95°C

[MISSING IMAGE: an02uf016, an02uf016 ]

Approximately 20  °C

8.24 min

After 10 minutes, the coffee is cooling at a rate of 2.5  °C/minute. The temperature is decreasing, so the rate of change will be negative.

A=30; T0=170

k=0.0697

T= 170e−0.0697t+30

[MISSING IMAGE: an02uf017, an02uf017 ]

89.8  °C

−4.2  °C/min

40.65 min

From the graph, the temperature of the metal ball is always greater than 30  °C, the temperature of the room, so if left in the room it will never reach a temperature of 10  °C.

Exercise 2.6 Review: exam practice

17

12

72

5

−2x

2x+4

2x+1

f ′(x)=2x+10

0; The function has a stationary point at x=−5.

10

5

0.792

3.084

−1.040

3.386

ln2

ln3

ln2

−ln2 ,ln4

−25

[MISSING IMAGE: an02uf018, an02uf018 ]

y=15x or y=(0.2)x

Domain: x∈R, range: y∈(1, ∞)

[MISSING IMAGE: an02uf019, an02uf019 ]

Domain: x∈R, range: y∈(−∞,3)

[MISSING IMAGE: an02uf020, an02uf020 ]

Domain: x∈R, range: y∈(−∞,0)

[MISSING IMAGE: an02uf021, an02uf021 ]

Domain: x∈R, range: y∈(−∞,−3)

[MISSING IMAGE: an02uf022, an02uf022 ]

Domain: x∈R, range: y∈(−4,∞)

[MISSING IMAGE: an02uf023, an02uf023 ]

Domain: x∈R, range: y∈(0,∞)

[MISSING IMAGE: an02uf024, an02uf024 ]

[MISSING IMAGE: an02uf025, an02uf025 ]

Domain R, range (–24, ∞)

[MISSING IMAGE: an02uf026, an02uf026 ]

A

−13e−13x

12x3+4xe−2x2

−83e−2x+43e−4x−2e−3x

4e4x−12e2x

12

a=−11, b=11; f:R→R, f(x)=−11ex+11

y= e−2x+4

5

dydx=−10xe−x2

3.89

−3.68

a=4; k= 0.059

dmdt=−0.236e−0.059t

0.17 g/h

A=250; k=0.047

506 000

2023

P0=500

16 000 bacteria.

At t=8, P′(8)=125ln2 bacteria/hour

16 hours

30

24 mg

k= 0.1116

17.17 mg

After 10.8 hours

$13 099.65

24 years and 5 months

A=28; T0=72

Sample responses can be found in the worked solutions in the online resources.

T=72e−13ln32t+28

[MISSING IMAGE: an02uf027, an02uf027 ]

60 °C

Since the room is kept at a constant temperature of 28  °C, the water will never cool below this level.